

Representations of skew group algebras induced from isomorphically invariant modules over path algebras*

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July 7, 2014

Abstract

Suppose that Q is a connected quiver without oriented cycles and σ is an automorphism of Q . Let k be an algebraically closed field whose characteristic does not divide the order of the cyclic group $\langle \sigma \rangle$.

The aim of this paper is to investigate the relationship between indecomposable kQ -modules and indecomposable $kQ \# k\langle \sigma \rangle$ -modules. It has been shown by Hubery that any $kQ \# k\langle \sigma \rangle$ -module is an isomorphically invariant kQ -module, i.e., ii-module (in this paper, we call it $\langle \sigma \rangle$ -equivalent kQ -module), and conversely any $\langle \sigma \rangle$ -equivalent kQ -module induces a $kQ \# k\langle \sigma \rangle$ -module. In this paper, the authors prove that a $kQ \# k\langle \sigma \rangle$ -module is indecomposable if and only if it is an indecomposable $\langle \sigma \rangle$ -equivalent kQ -module. Namely, a method is given in order to induce all indecomposable $kQ \# k\langle \sigma \rangle$ -modules from all indecomposable $\langle \sigma \rangle$ -equivalent kQ -modules. The number of non-isomorphic indecomposable $kQ \# k\langle \sigma \rangle$ -modules induced from the same indecomposable $\langle \sigma \rangle$ -equivalent kQ -module is given. In particular, the authors give the relationship between indecomposable $kQ \# k\langle \sigma \rangle$ -modules and indecomposable kQ -modules in the cases of indecomposable simple, projective and injective modules.

1 Introduction

There is a lot of literature on smash product algebras $A \# H$ and crossed product algebras $A \#_{\sigma} H$, and on their relationships with the algebra A^H , whose elements are those elements

*Project supported by the Program for New Century Excellent Talents in University (No.04-0522) and the National Natural Science Foundation of China (No.10571153)

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of A left fixed by H . Much work has been done to determine which properties of A are inherited by $A\#_{\sigma}H$. The works about the relationships among these algebras were motivated by the development of the Galois theory of noncommutative algebras.

It is important to totally understand the relationships among representations of A , $A\#H$ and $A\#_{\sigma}H$. It has been proven in [8, 12] that the representation types of A , $A\#H$ and $A\#_{\sigma}H$ are the same if A is a finite dimensional algebra and H is a finite dimensional semisimple and cosemisimple Hopf algebra over an algebraically closed field. These results were used to classify finite dimensional basic Hopf algebras through their representation types in [8, 9, 10, 11]. In particular, each $A\#H$ -module (or respectively, $A\#_{\sigma}H$ -module) is an A -module, but not all A -modules induce $A\#H$ -modules (or respectively, $A\#_{\sigma}H$ -modules). Thus, some questions arise, such as the following examples:

- (i) What kind of A -modules can induce $A\#H$ -modules (or respectively, $A\#_{\sigma}H$ -modules)?
- (ii) If an A -module can induce $A\#H$ -modules (or respectively, $A\#_{\sigma}H$ -modules), how many non-isomorphic classes of such induced $A\#H$ -modules (or respectively, $A\#_{\sigma}H$ -modules) exist?

In [8, 12], we have proven that for finite dimensional algebra A , finite dimensional Hopf algebra H such that H and its dual H^* are both semisimple, then A , $A\#H$ and $A\#_{\sigma}H$ have the same representation type. This result allows us the possibility to discuss these questions.

Hubery in [5, 6] constructed the dual quiver with automorphism $(\tilde{Q}, \tilde{\sigma})$, where \tilde{Q} is the Ext-quiver of $kQ\#k\langle\sigma\rangle$ and $\tilde{\sigma}$ is the automorphism of $k\tilde{Q}$ induced from an admissible automorphism σ . Here, the admissible automorphism σ means that Q has no arrow connecting two vertices in the same σ -orbit, and k is an algebraically closed field of characteristic not dividing the order of $\langle\sigma\rangle$. Hubery used the dual quiver $(\tilde{Q}, \tilde{\sigma})$ to prove the generalization of Kac's Theorem. During the construction, Hubery defined the isomorphically invariant module, i.e. ii-module (in this paper, we call it $\langle\sigma\rangle$ -equivalent kQ -module), and proved that any $kQ\#k\langle\sigma\rangle$ -module is an ii-module and conversely any ii-module induces a $kQ\#k\langle\sigma\rangle$ -module. These works by Hubery are crucial for us to answer the above questions in the special case when A is a path algebra kQ , and $H = k\langle\sigma\rangle$, a cyclic group algebra with $\sigma \in \text{Aut}(Q)$.

In this paper, we investigate the relationship between indecomposable modules over the path algebra kQ and the skew group algebra $kQ\#k\langle\sigma\rangle$ respectively, where k is an algebraically closed field with the characteristic not dividing the order of σ , Q is connected and without oriented cycles, and $\sigma \in \text{Aut}(Q)$. We prove that a $kQ\#k\langle\sigma\rangle$ -module is indecomposable if and only if it is an indecomposable $\langle\sigma\rangle$ -equivalent kQ -module. Namely,

a method is given in order to induce all indecomposable $kQ \# k\langle\sigma\rangle$ -modules from each indecomposable $\langle\sigma\rangle$ -equivalent kQ -module. The number of non-isomorphic indecomposable $kQ \# k\langle\sigma\rangle$ -modules induced from the same indecomposable $\langle\sigma\rangle$ -equivalent kQ -module is given.

In this paper, assume that all the modules are unital and finitely generated and that k is always an algebraically closed field. All the concepts and notations on Hopf algebra and crossed product algebra can be found in [13]. We fix the notation $\Delta(h) = \sum_{(h)} h_1 \otimes h_2$ under the Sweedler meaning. If H is a group algebra kG , $\Delta(g) = g \otimes g$, $\varepsilon(g) = 1$, for all $g \in G$. In particular, we give the definition of smash product as follows:

Definition 1.1 *Let H be a Hopf algebra. An algebra A is a (left) H -module algebra if for all $h \in H, a, b \in A$,*

- (1) *A is a (left) H -module, via $h \otimes a \mapsto h \cdot a$,*
- (2) *$h \cdot (ab) = \sum_{(h)} (h_1 \cdot a)(h_2 \cdot b)$,*
- (3) *$h \cdot 1_A = \varepsilon(h)1_A$.*

Definition 1.2 *Let A be a left H -module algebra. The smash product algebra $A \# H$ is defined by satisfying that*

- (1) *as a k -space, $A \# H = A \otimes H$;*
- (2) *the multiplication is given by $(a \# h)(b \# l) = \sum_{(h)} a(h_1 \cdot b) \# h_2 l$ for all $a, b \in A, h, l \in H$.*

We write that by $a \# h$ the element $a \otimes h \in A \# H$.

2 Isomorphically Invariant kQ -modules

Suppose that $Q = (Q_0, Q_1)$ is a *quiver* given by the *vertex set* Q_0 and the *arrow set* Q_1 . For an arrow $\alpha \in Q_1$, the vertex $s(\alpha)$ is the *start vertex* of α and the vertex $t(\alpha)$ is the *end vertex* of α , and we draw $s(\alpha) \xrightarrow{\alpha} t(\alpha)$. A *path* in Q is $(b|\alpha_n \cdots \alpha_1|a)$, where $\alpha_i \in Q_1$, for $i = 1, \dots, n$, and $s(\alpha_1) = a, t(\alpha_i) = s(\alpha_{i+1})$, for $i = 1, \dots, n-1$, and $t(\alpha_n) = b$. The *length of a path* is the number of arrows in it. To each arrow α we can assign an edge $\overline{\alpha}$ where the orientation is forgotten. A *walk* between two vertices a and b is given by $(b|\overline{\alpha_n} \cdots \overline{\alpha_1}|a)$, where $a \in \{s(\alpha_1), t(\alpha_1)\}, b \in \{s(\alpha_n), t(\alpha_n)\}$, and for each $i = 1, \dots, n-1$, $\{s(\alpha_i), t(\alpha_i)\} \cap \{s(\alpha_{i+1}), t(\alpha_{i+1})\} \neq \emptyset$. A quiver is said to be *connected* if for each pair of vertices a and b , there exists a walk between them.

Denote by $\text{Rep}Q$ the category of representations of the quiver Q over k .

It is well-known that a *representation* $X = (X_i, i \in Q_0; X_\rho : X_{s(\rho)} \rightarrow X_{t(\rho)}, \rho \in Q_1)$ of Q is given by finite dimensional k -vector spaces X_i for all $i \in Q_0$ and k -linear maps $X_\rho : X_{s(\rho)} \rightarrow X_{t(\rho)}$ for all arrows $\rho \in Q_1$; a *morphism* $\theta : X \rightarrow X'$ is given by k -linear maps $\theta_i : X_i \rightarrow X'_i$ for $i \in Q_0$ satisfying $X'_\rho \theta_{s(\rho)} = \theta_{t(\rho)} X_\rho$ for all $\rho \in Q_1$. The composition of θ with another morphism $\phi : X' \rightarrow X''$ is defined by $(\phi\theta)_i = \phi_i \theta_i$ for all $i \in Q_0$.

Let Q be a finite quiver, that is, $|Q_0|$ and $|Q_1|$ are both finite. Denote by $\text{mod}kQ$ the category of finite generated kQ -modules. All through the paper, Q is a connected quiver without oriented cycles.

For a kQ -module \mathcal{X} , define a representation X with the k -vector spaces $X_i = e_i \mathcal{X}$ for all vertices $i \in Q_0$ and the linear maps X_ρ for all arrows $\rho \in Q_1$ satisfying $X_\rho(x) = \rho x = e_{t(\rho)} \rho x \in X_{t(\rho)}$ for $x \in X_{s(\rho)}$. Conversely, for a representation X of Q , define a kQ -module \mathcal{X} via $\mathcal{X} = \bigoplus_{i \in Q_0} X_i$ with actions of paths $\rho_1 \cdots \rho_m$ satisfying $\rho_1 \cdots \rho_m x = \varepsilon_{t(\rho_1)} X_{\rho_1} \cdots X_{\rho_m} \pi_{s(\rho_m)}(x)$ and $e_i x = \varepsilon_i \pi_i(x)$ for the canonical maps $X_i \xrightarrow{\varepsilon_i} \mathcal{X} \xrightarrow{\pi_i} X_i$. Then, as we have well-known, this correspondence gives a pair of mutually quasi-invertible functors between $\text{Rep}Q$ and $\text{mod}kQ$, that is,

Theorem 2.1 ([1, 2]) *For a finite quiver Q over a field k , the categories $\text{Rep}Q$ and $\text{mod}kQ$ are equivalent.*

The correspondence, given above between objects of $\text{Rep}Q$ and $\text{mod}kQ$, will be useful for our further discussion.

From now on, we let Q be a connected quiver and without oriented cycles, $\sigma \in \text{Aut}(Q)$ and k is an algebraically closed field with the characteristic of not dividing the order of σ .

It is easy to extend σ linearly to the whole k -linear space kQ as a k -automorphism, i.e., $\sigma \in \text{Aut}_k kQ$.

Let \mathcal{X} be a kQ -module. We define a kQ -module ${}^\sigma \mathcal{X}$ by taking the same underlying vector space as \mathcal{X} but with the new action:

$$p \cdot x := \sigma^{-1}(p)x \quad \text{for } p \in kQ.$$

Let $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ be a module homomorphism, and set ${}^\sigma \phi = \phi$ as a linear map. Then,

$$\phi(p \cdot x) = \phi(\sigma^{-1}(p)x) = \sigma^{-1}(p)\phi(x) = p \cdot \phi(x)$$

which means ${}^\sigma \phi : {}^\sigma \mathcal{X} \rightarrow {}^\sigma \mathcal{Y}$ is a homomorphism of modules under the new module action.

Let $X = (X_i, i \in Q_0; X_\rho : X_{s(\rho)} \rightarrow X_{t(\rho)}, \rho \in Q_1)$ be a representation of Q and \mathcal{X} with the corresponding kQ -module via the functor described in Theorem 2.1, so $\mathcal{X} = \bigoplus_{i \in Q_0} X_i$. We describe the representation ${}^\sigma X = ({}^\sigma X_i, i \in Q_0; {}^\sigma X_\rho : {}^\sigma X_{s(\rho)} \rightarrow {}^\sigma X_{t(\rho)}, \rho \in Q_1)$ corresponding to the module ${}^\sigma \mathcal{X}$ in terms of the original representation X .

Proposition 2.2 *For a representation $X = (X_i, i \in Q_0; X_\rho : X_{s(\rho)} \rightarrow X_{t(\rho)}, \rho \in Q_1)$ of a quiver Q and \mathcal{X} with the corresponding kQ -module, the corresponding representation ${}^\sigma X$ of the module ${}^\sigma \mathcal{X}$,*

$${}^\sigma X = ({}^\sigma X_i, i \in Q_0, {}^\sigma X_\rho : {}^\sigma X_{s(\rho)} \rightarrow {}^\sigma X_{t(\rho)}, \rho \in Q_1),$$

is given with ${}^\sigma X_i = X_{\sigma^{-1}(i)}$ as vector spaces, and the map ${}^\sigma X_\rho : {}^\sigma X_{s(\rho)} \rightarrow {}^\sigma X_{t(\rho)}$ is the same as $X_{\sigma^{-1}(\rho)} : X_{\sigma^{-1}(s(\rho))} \rightarrow X_{\sigma^{-1}(t(\rho))}$.

Proof. For all $j \in Q_0$ and $\rho \in Q_1, x \in {}^\sigma X_{s(\rho)} = X_{\sigma^{-1}(s(\rho))}$, we have that

$${}^\sigma X_j = e_j \cdot {}^\sigma \mathcal{X} = \sigma^{-1}(e_j) {}^\sigma \mathcal{X} = e_{\sigma^{-1}(j)} (\bigoplus_{i \in Q_0} X_i) = X_{\sigma^{-1}(j)};$$

$$\begin{aligned} {}^\sigma X_\rho(x) &= \rho \cdot x = \rho \cdot (e_{s(\rho)} \cdot (\delta_{\sigma^{-1}(s(\rho))} i x)_{i \in Q_0}) = \sigma^{-1}(\rho) (e_{\sigma^{-1}(s(\rho))} (\delta_{\sigma^{-1}(s(\rho))} i x)_{i \in Q_0}) \\ &= \sigma^{-1}(\rho)(x) = X_{\sigma^{-1}(\rho)}(x) \in X_{\sigma^{-1}(t(\rho))} = {}^\sigma X_{t(\rho)}. \blacksquare \end{aligned}$$

Thus, let $\varphi = (\varphi_i)_{i \in Q_0} : X \rightarrow Y$ be the morphism between two representations, then ${}^\sigma \varphi = ({}^\sigma \varphi_i)_{i \in Q_0} : {}^\sigma X \rightarrow {}^\sigma Y$ satisfies ${}^\sigma \varphi_i = \varphi_{\sigma^{-1}(i)}$ as a linear map.

In this way, we obtain an additive equivalence functor $F(\sigma)$, with inverse $F(\sigma^{-1})$, on $\text{mod} kQ$ (or say, on $\text{Rep}(Q)$), which send \mathcal{X} (or say, on X) to ${}^\sigma \mathcal{X}$ (or say, to ${}^\sigma X$) and send $\phi : \mathcal{X} \rightarrow \mathcal{Y}$ (or say, $\varphi : X \rightarrow Y$) to ${}^\sigma \phi : {}^\sigma \mathcal{X} \rightarrow {}^\sigma \mathcal{Y}$ (or say, ${}^\sigma \varphi : {}^\sigma X \rightarrow {}^\sigma Y$), and satisfies that $F(\sigma^r) = F(\sigma)^r$ for any integer r .

Of course, \mathcal{X} (or say, X) is indecomposable if and only if ${}^\sigma \mathcal{X}$ (or say, ${}^\sigma X$) is so.

In the sequel, we always identify objects and morphisms of $\text{mod} kQ$ with the corresponding ones of $\text{Rep} Q$.

We call a representation X of a quiver Q *isomorphically invariant* by $\langle \sigma \rangle$ (or say, $\langle \sigma \rangle$ -*equivalent*) if there is a representation isomorphism ${}^\sigma X \cong X$. Equivalently, we can define an *isomorphically invariant module* (or say, $\langle \sigma \rangle$ -*equivalent module*).

Let $|Q_0| = s$, Z the set of all integers. Then, we can define $\sigma : Z^s \rightarrow Z^s$ by $\sigma(\underline{\dim} X) = \underline{\dim} {}^\sigma X$. So, a $\langle \sigma \rangle$ -equivalent representation has a dimension vector fixed by this σ .

Lemma 2.3 ([5, 6]) *Any indecomposable $\langle \sigma \rangle$ -equivalent representation X in $\text{Rep} Q$ is precisely the representation of the form*

$$X \cong Y \oplus {}^\sigma Y \oplus \dots \oplus {}^{\sigma^{m-1}} Y$$

where Y is an indecomposable Q -representation and $m \geq 1$ is the minimal integer such that ${}^{\sigma^m} Y \cong Y$. Moreover, the Krull-Remak-Schmidt Theorem holds for $\langle \sigma \rangle$ -equivalent representations.

We call X an *indecomposable $\langle\sigma\rangle$ -equivalent representation* if it is not isomorphic to the proper direct sum of two $\langle\sigma\rangle$ -equivalent representations. Lemma 2.3 means that any $\langle\sigma\rangle$ -equivalent representation is a direct sum of indecomposable $\langle\sigma\rangle$ -equivalent representations. The following lemma tells us the relation between m and n .

Lemma 2.4 *For any indecomposable $\langle\sigma\rangle$ -equivalent kQ -module $Y \oplus \sigma Y \oplus \dots \oplus \sigma^{m-1}Y$, with Y an indecomposable kQ -module and m the minimal integer such that $\sigma^m Y \cong Y$. Then $r = n/m$ is an integer.*

Proof. From $\sigma^m Y \cong Y$, we have that $\sigma^{km+l} X \cong \sigma^l X$, for any non-negative integer k and $l \in \{0, 1, \dots, m-1\}$. And from $\sigma^n X = X$ and the minimality of m such that $\sigma^m X \cong X$, we have that there exists an integer r such that $rm = n$. ■

Example 2.1 *Let Q be the quiver $-1 \xrightarrow{\alpha} 0 \xleftarrow{\beta} 1$ and $\sigma \in \text{Aut}(Q)$, which is defined as $\sigma(e_{-1}) = e_1, \sigma(e_0) = e_0, \sigma(e_1) = e_{-1}, \sigma(\alpha) = \beta, \sigma(\beta) = \alpha$. All indecomposable kQ -representations are:*

$$\begin{aligned} L_{-1} : k \rightarrow 0 \leftarrow 0; \quad L_0 : 0 \rightarrow k \leftarrow 0; \quad L_1 : 0 \rightarrow 0 \leftarrow k; \\ L_{-10} : k \xrightarrow{1} k \leftarrow 0; \quad L_{01} : 0 \rightarrow k \xleftarrow{1} k; \quad L_{101} : k \xrightarrow{1} k \xleftarrow{1} k. \end{aligned}$$

By Proposition 2.2, we have ${}^\sigma L_{-1} = L_1, {}^\sigma L_1 = L_{-1}, {}^\sigma L_{-10} = L_{01}, {}^\sigma L_{01} = L_{-10}, {}^\sigma L_0 = L_0, {}^\sigma L_{101} = L_{101}$. By definitions, all indecomposable $\langle\sigma\rangle$ -equivalent representations are:

$$L_1 \oplus L_{-1}; \quad L_{01} \oplus L_{-10}; \quad L_0; \quad L_{101}.$$

3 Structure of modules over a skew group algebra from a cyclic group

In this section, we denote by (Q, σ) a fixed connected finite quiver Q without oriented cycle and a quiver automorphism $\sigma \in \text{Aut}Q$ of order n . Then, we have a cyclic group $\langle\sigma\rangle$ of order n and a skew group algebra $kQ \# k\langle\sigma\rangle$. In the sequel, we will always assume that k is an algebraically closed field with the characteristic of not dividing n .

The following lemma can be found in [5], but we still give its proof because it is useful for our discussion.

Lemma 3.1 ([5, 6]) *Every module X of the skew group algebra $kQ \# k\langle\sigma\rangle$ is a $\langle\sigma\rangle$ -equivalent kQ -module.*

Proof. In order to show X to be a $\langle \sigma \rangle$ -equivalent kQ -module, we only need to prove $X \cong {}^\sigma X$ as kQ -modules. Define $f : {}^\sigma X \rightarrow X$ such that $f(x) = \sigma x$, for all $x \in X$. It is well-defined since X is a $kQ \# k\langle \sigma \rangle$ -module. We have that for any $p \in kQ$,

$$f(p \cdot x) = \sigma(p \cdot x) = \sigma(\sigma^{-1}(p)x) = \sigma(\sigma^{-1}(p))x = (p \# \sigma)x = p(\sigma x) = pf(x),$$

which means that f is a kQ -module homomorphism. Moreover, f is an isomorphism with inverse $f^{-1} : X \rightarrow {}^\sigma X$ such that $f^{-1}(x) = \sigma^{-1}x$ for all $x \in X$. ■

By Lemma 3.1 and Lemma 2.3, we have for any $kQ \# k\langle \sigma \rangle$ -module X , $X \cong_{i=1}^s Y_i$, where $Y_i \cong X_i \oplus {}^\sigma X_i \oplus \cdots \oplus \sigma^{m_i-1} X_i$ with X_i an indecomposable kQ -module and m_i a minimal positive integer such that $\sigma^{m_i} X_i \cong X_i$. Hence we have the following kQ -isomorphism, say g :

$$X \xrightarrow{g} \bigoplus_{i=1}^s \bigoplus_{j=0}^{m_i-1} \sigma^j X_i.$$

Then we can define the $kQ \# k\langle \sigma \rangle$ -module structure on $\bigoplus_{i=1}^s \bigoplus_{j=0}^{m_i-1} \sigma^j X_i$ through g . In fact, we define σ 's action on $\bigoplus_{i=1}^s \bigoplus_{j=0}^{m_i-1} \sigma^j X_i$ by

$$\sigma y = g(\sigma g^{-1}(y)),$$

and $(p\sigma^l)y = p(\sigma^l y)$ for any $p \in kQ$, $y \in \bigoplus_{i=1}^s \bigoplus_{j=0}^{m_i-1} \sigma^j X_i$, $l = 0, \dots, n-1$.

The action makes $\bigoplus_{i=1}^s \bigoplus_{j=0}^{m_i-1} \sigma^j X_i$ a $kQ \# k\langle \sigma \rangle$ -module since by definition of the smash product, $\sigma p = \sigma(p) \# \sigma$, then

$$\begin{aligned} \sigma(py) &= g(\sigma(g^{-1}(py))) = g(\sigma(pg^{-1}(y))) = g(\sigma(p)(\sigma(g^{-1}y))) \\ &= \sigma(p)(g(\sigma(g^{-1}y))) = \sigma(p)(\sigma y) = (\sigma(p)\sigma)y \\ &= (\sigma p)y. \end{aligned}$$

Moreover, g is a $kQ \# k\langle \sigma \rangle$ -module homomorphism via $g(\sigma x) = g(\sigma(g^{-1}(g(x)))) = \sigma(g(x))$ for any $x \in X$.

Considering the restriction of σ on each indecomposable kQ -module X_i , we have:

Corollary 3.2 *With the above notations, $\sigma^j X_i \cong \sigma^j X_i$ as kQ -modules for any $j \in \{1, \dots, m_i\}$, $i \in \{1, \dots, s\}$.*

Proof. Define $f : \sigma^j X_i \rightarrow \sigma^j X_i$ by $f(x) = \sigma^j x = g(\sigma^j(g^{-1}(x)))$ for any $x \in \sigma^j X_i$. Then, for any $p \in kQ$,

$$\begin{aligned} f(p \cdot x) &= \sigma^j(p \cdot x) = g(\sigma^j(g^{-1}(p \cdot x))) = g(\sigma^j(g^{-1}(\sigma^{-j}(p)x))) \\ &= g(g^{-1}(\sigma^j(\sigma^{-j}(p)x))) = \sigma^j(\sigma^{-j}(p)x) = (p \# \sigma^j)x = p(\sigma^j(x)) \\ &= p(f(x)), \end{aligned}$$

thus f is a kQ -module isomorphism with inverse f^{-1} satisfying $f^{-1}(y) = \sigma^{-j}y$ for any $y \in \sigma^j X_i$. ■

By corollary 3.2, for any $kQ\#k\langle\sigma\rangle$ -module X , $X \cong \oplus_{i=1}^s Y_i$ with Y_i as an indecomposable $\langle\sigma\rangle$ -equivalent kQ -module, σ 's action may be closed in each Y_i . Interestingly, whether σ 's action is closed in each Y_i is the same thing as whether any indecomposable $kQ\#k\langle\sigma\rangle$ -module is an indecomposable $\langle\sigma\rangle$ -equivalent kQ -module.

From any indecomposable kQ -module X , we get an indecomposable $\langle\sigma\rangle$ -equivalent kQ -module $\oplus_{i=1}^{m-1} \sigma^i X$, and then we have several questions to solve:

Question 1. Is it possible to endow $\oplus_{i=1}^{m-1} \sigma^i X$ with an induced $kQ\#k\langle\sigma\rangle$ -module structure?

Question 2. If the kQ -module $\oplus_{i=1}^{m-1} \sigma^i X$ can be endowed with some $kQ\#k\langle\sigma\rangle$ -module structures, how many non-isomorphic classes of such induced $kQ\#k\langle\sigma\rangle$ -modules exist?

4 Construction of indecomposable $kQ\#k\langle\sigma\rangle$ -modules from indecomposable kQ -modules

It is known in [5, 6] that any $\langle\sigma\rangle$ -equivalent kQ -module X induces a $kQ\#k\langle\sigma\rangle$ -module. For completeness, we give its proof below:

Proposition 4.1 ([5, 6]) *Let X be a $\langle\sigma\rangle$ -equivalent kQ -module. Then there exists an isomorphism $\phi : {}^\sigma X \rightarrow X$ such that $\phi^n = \phi \sigma \phi \cdots \sigma^{n-1} \phi$ of X is the identity.*

Theorem 4.2 ([5, 6]) *Let X be a $\langle\sigma\rangle$ -equivalent kQ -module. Then X has an induced $kQ\#k\langle\sigma\rangle$ -module structure if we define $\sigma^i(x) = \phi^i(x)$.*

Proof. Since for $p\#\sigma^i, q\#\sigma^j \in kQ\#k\langle\sigma\rangle, x \in X$,

$$\begin{aligned} (p\#\sigma^i)((q\#\sigma^j)(x)) &= (p\#\sigma^i)(q\phi^j(x)) = p\phi^i(q\phi^j(x)) = p\phi^{i-1}\phi(\sigma(q) \cdot \phi^j(x)) \\ &= p\phi^{i-1}(\sigma(q)\phi^{j+1}(x)) = \cdots = p\sigma^i(q)(\phi^{i+j}(x)) \\ &= (p\sigma^i(q)\#\sigma^{i+j})(x) = (p\#\sigma^i)(q\#\sigma^j)(x). \quad \blacksquare \end{aligned}$$

This result means Question 1 can be answered affirmatively.

The induced $kQ\#k\langle\sigma\rangle$ -module constructed in Theorem 4.2 will be used accordingly in our conclusions below. So, we will call such induced module a *canonical induced $kQ\#k\langle\sigma\rangle$ -module*.

Again, we recall two lemmas, which will be used in the next two sections.

Lemma 4.3 ([14]) *Let X, Y be indecomposable kQ -modules, and G be a subgroup of the k -automorphism group of kQ . Then:*

- (i) $(kQ \# kG) \otimes_{kQ} X \cong \bigoplus_{g \in G} {}^g X$ as kQ -modules;
- (ii) $(kQ \# kG) \otimes_{kQ} X \cong (kQ \# kG) \otimes_{kQ} Y$ if and only if $Y \cong {}^g X$ for some $g \in G$;
- (iii) The number of summands in the decomposition of $(kQ \# kG) \otimes_{kQ} X$ into a direct sum of indecomposables is at most the order of H , where $H = \{g \in G, {}^g X \cong X\}$;
- (iv) If G is cyclic of order n and $X \cong {}^g X$ for all $g \in G$, then $(kQ \# kG) \otimes_{kQ} X$ has exactly n summands;
- (v) If $H = \{g \in G, {}^g X \cong X\}$ is cyclic of order m , then $(kQ \# kG) \otimes_{kQ} X$ has exactly m summands.

Lemma 4.4 ([8, 12]) *Let H be a finite dimensional semisimple Hopf algebra and A be a finite dimensional H -module algebra. Then, for any $A \# H$ -module X , it holds that $X \mid (A \# H) \otimes_A X$, that is, X is a direct summand of $(A \# H) \otimes_A X$ as an $A \# H$ -module.*

4.1 Induction of indecomposable $kQ \# k\langle \sigma^m \rangle$ -modules from an indecomposable kQ -module X with minimal m satisfying $\sigma^m X \cong X$

For any indecomposable kQ -module X with minimal m such that $\sigma^m X \cong X$, let $L = \{g \in \langle \sigma \rangle \mid {}^g X \cong X\}$, then $L = \langle \sigma^m \rangle$, a cyclic group generated by σ^m by Lemma 2.4. Since $kL \cong k\langle \sigma^m \rangle$ is a semisimple group algebra, we have

$$k\langle \sigma^m \rangle \cong \bigoplus_{i=1}^{r=n/m} L_i \quad (1)$$

as a $k\langle \sigma^m \rangle$ -module, where L_i is isomorphic to k as a vector space, and $\sigma^{m'}$'s action is $\sigma^m(1) = \zeta^i$, where ζ is the r -th primitive root of 1. Moreover, $L_i \not\cong L_j$ as $k\langle \sigma^m \rangle$ -modules, if $i \neq j$.

Before answering Question 2, we introduce the following question:

Question 3. For any indecomposable kQ -module X with minimal m such that $\sigma^m X \cong X$, how many non-isomorphic indecomposable $kQ \# k\langle \sigma^m \rangle$ -modules can be induced from X ?

Since X is a $\langle \sigma^m \rangle$ -equivalent kQ -module, by Proposition 4.1 and Theorem 4.2, there exists ϕ_X , such that X induces a $kQ \# k\langle \sigma^m \rangle$ -module structure. Using the $kQ \# k\langle \sigma^m \rangle$ -module structure on X , we can define $kQ \# k\langle \sigma^m \rangle$ -module structure on $L_i \otimes_k X$, for any $i \in \{1, \dots, r\}$. In fact, for any $i, j \in \{1, 2, \dots, r\}$, $1 \otimes x \in L_i \otimes_k X, p \in kQ$, define

$$p \# \sigma^{mj}(1 \otimes x) = \sigma^{mj}(1) \otimes p \# l(x) = \zeta^{ij} \otimes p \# l(x),$$

where the action $p\#l(x)$ is inherited from the canonical induced $kQ\#k\langle\sigma^m\rangle$ -module structure. Then

Lemma 4.5 *With the above notations, we have:*

- (i) $L_i \otimes_k X \cong X$ as kQ -modules for any $i \in \{1, 2, \dots, r\}$;
- (ii) $L_i \otimes_k X$ is an indecomposable $kQ\#k\langle\sigma^m\rangle$ -module for any $i \in \{1, 2, \dots, r\}$;
- (iii) $L_i \otimes_k X \not\cong L_j \otimes_k X$ as $kQ\#k\langle\sigma^m\rangle$ -modules, if $i \neq j$.

Proof. (i) Define $f : X \rightarrow L_i \otimes_k X$ by $x \mapsto 1 \otimes x$, then f is a kQ -module homomorphism since $f(p(x)) = 1 \otimes p(x) = p(1 \otimes x) = pf(x)$, for any $p \in kQ, x \in X$. Obviously, f is bijective.

(ii) $L_i \otimes_k X$ is $kQ\#k\langle\sigma^m\rangle$ -module, since for any $p_1, p_2 \in kQ, j_1, j_2 \in \{1, 2, \dots, r\}, x \in X$,

$$\begin{aligned}
 (1\#1)(1 \otimes x) &= \sigma^{mr}(1) \otimes 1\#1(x) = \zeta^{ir} \otimes x = 1 \otimes x, \\
 ((p_2\#\sigma^{mj_2})(p_1\#\sigma^{mj_1}))(1 \otimes x) &= (p_2\sigma^{mj_2}(p_1)\#\sigma^{mj_2}\sigma^{mj_1})(1 \otimes x) \\
 &= (p_2\sigma^{mj_2}(p_1)\#\sigma^{m(j_2+j_1)})(1 \otimes x) \\
 &= \sigma^{m(j_2+j_1)}(1) \otimes (p_2\sigma^{mj_2}(p_1)\#\sigma^{mj_2}\sigma^{mj_1})(x) \\
 &= \zeta^{i(j_2+j_1)}(1) \otimes (p_2\sigma^{mj_2}(p_1)\#\sigma^{mj_2}\sigma^{mj_1})(x) \\
 &= \zeta^{ij_2}\zeta^{ij_1}(1) \otimes (p_2\#\sigma^{mj_2})(p_1\#\sigma^{mj_1})(x) \\
 &= p_2\#\sigma^{mj_2}(\zeta^{ij_1}(1) \otimes p_1\#\sigma^{mj_1}(x)) \\
 &= p_2\#\sigma^{mj_2}(p_1\#\sigma^{mj_1}(1 \otimes x)).
 \end{aligned}$$

And $L_i \otimes_k X$ is an indecomposable $kQ\#k\langle\sigma^m\rangle$ -module since it is an indecomposable kQ -module by (i).

Before giving a proof of (iii), we need to perform some preparations as follows.

Lemma 4.6 *For any $i \in \{1, 2, \dots, r\}$, $\text{Hom}_{kQ}(X, L_i \otimes_k X) \cong L_i \otimes_k \text{End}_{kQ}(X)$ as $kQ\#k\langle\sigma^m\rangle$ -modules.*

Proof. The $kQ\#k\langle\sigma^m\rangle$ -module structure of $\text{Hom}_{kQ}(X, L_i \otimes_k X)$ is given by

$$(p\#l(f))(x) = (p\#l)f(x), \forall f \in \text{Hom}_{kQ}(X, L_i \otimes_k X), p\#l \in kQ\#k\langle\sigma^m\rangle, x \in X;$$

The $kQ\#k\langle\sigma^m\rangle$ -module structure of $\text{End}_{kQ}(X)$ is given by

$$(p\#l(f))(x) = (p\#l)f(x), \forall f \in \text{End}_{kQ}(X), p\#l \in kQ\#k\langle\sigma^m\rangle, x \in X;$$

The $kQ\#k\langle\sigma^m\rangle$ -module structure of $L_i \otimes_k \text{End}_{kQ}(X)$ is given by

$$(p\#l)(1\#f) = l(1)\#(p\#l)(f), \forall f \in \text{End}_{kQ}(X), p\#l \in kQ\#k\langle\sigma^m\rangle.$$

Define $F : \text{Hom}_{kQ}(X, L_i \otimes_k X) \rightarrow L_i \otimes_k \text{End}_{kQ}(X)$ by

$$F(f) = 1 \otimes \bar{f}, \forall f \in \text{Hom}_{kQ}(X, L_i \otimes_k X),$$

where \bar{f} is defined by $\bar{f}(x) = k_f x_f$, if $f(x) = k_f \otimes x_f, \forall x \in X$. Since for any $p \in kQ, x \in X$, $f(px) = pf(x) = p(k_f \otimes x_f) = k_f \otimes p(x_f) = 1 \otimes p(k_f x_f)$, then $\bar{f}(px) = k_f p(x_f) = p(k_f x_f) = p\bar{f}(x)$, i.e., $\bar{f} \in \text{End}_{kQ}(X)$, which means F is well-defined.

Show F is a $kQ\#k\langle\sigma\rangle$ -module homomorphism. Since for any $p \in kQ, f \in \text{Hom}_{kQ}(X, L_i \otimes_k X)$, $(pf)(x) = p(f(x)) = p(k_f \otimes x_f) = k_f \otimes p(x)$, then $F(pf) = 1 \otimes \overline{pf} = 1 \otimes p\bar{f} = pF(f)$, which means F is a kQ -module homomorphism. And since $(\sigma^m f)(x) = \sigma^m(f(x)) = \sigma^m(k_f \otimes x_f) = \zeta^i k_f \otimes \sigma^m x_f$, then $F(\sigma^m f) = 1 \otimes \overline{\sigma^m f} = 1 \otimes \zeta^i \sigma^m \bar{f} = \zeta^i \otimes \sigma^m \bar{f} = \sigma^m(1 \otimes \bar{f})$, which means F is a $k\langle\sigma^m\rangle$ -module homomorphism.

Finally, F is a $kQ\#k\langle\sigma^m\rangle$ -module isomorphism, since F is injective and

$$\dim_k \text{Hom}_{kQ}(X, L_i \otimes_k X) = \dim_k L_i \otimes_k \text{End}_{kQ}(X). \blacksquare$$

Now we go back to the proof of (iii):

Otherwise, $L_i \otimes_k X \cong L_j \otimes_k X$ as $kQ\#k\langle\sigma^m\rangle$ -modules, for some $i \neq j$, then by Lemma 4.6, $L_i \otimes_k \text{End}_{kQ}(X) \cong L_j \otimes_k \text{End}_{kQ}(X)$ as $kQ\#k\langle\sigma^m\rangle$ -modules. Since $\text{End}_{kQ}(X)$ is a local ring, k is algebraically closed, we have $\text{End}_{kQ}(X)/\text{radEnd}_{kQ}(X) \cong k$ as algebras. Since $\text{radEnd}_{kQ}(X)$ is closed under $k\langle\sigma^m\rangle$ -module structure, we have $L_i \otimes_k \text{End}_{kQ}(X)/\text{radEnd}_{kQ}(X) \cong L_j \otimes_k \text{End}_{kQ}(X)/\text{radEnd}_{kQ}(X)$, which induces $L_i \cong L_j$ as $k\langle\sigma^m\rangle$ -modules, with contradiction to $i \neq j$. \blacksquare

Theorem 4.7 *Let X be an indecomposable kQ -module with m minimal such that $\sigma^m X \cong X$, L_i is defined in the isomorphism relation (1), for any $i \in \{1, 2, \dots, r\}$. Then the following statements hold:*

- (i) $(kQ\#k\langle\sigma^m\rangle) \otimes_{kQ} X$ is isomorphic to the direct sum of r non-isomorphic $kQ\#k\langle\sigma^m\rangle$ -modules, that is, $(kQ\#k\langle\sigma^m\rangle) \otimes_{kQ} X \cong \bigoplus_{i=1}^r L_i \otimes_k X$ as $kQ\#k\langle\sigma^m\rangle$ -modules;
- (ii) For any $kQ\#k\langle\sigma^m\rangle$ -module Y , if $Y \cong X$ as kQ -modules, then there exists a unique $i \in \{1, 2, \dots, r\}$, such that $Y \cong L_i \otimes_k X$. That is, there are r non-isomorphic $kQ\#k\langle\sigma^m\rangle$ -modules induced from X .

Proof. (i) For any $i \in \{1, 2, \dots, r\}$, since $L_i \otimes_k X \mid kQ\#k\langle\sigma^m\rangle \otimes_{kQ} (L_i \otimes_k X)$ by Lemma 4.4, $kQ\#k\langle\sigma^m\rangle \otimes_{kQ} (L_i \otimes_k X) \cong kQ\#k\langle\sigma^m\rangle \otimes_{kQ} X$ by Lemma 4.5(i), then we have $L_i \otimes_k X \mid$

$kQ\#k\langle\sigma^m\rangle \otimes_{kQ} X$. Then by Lemma 4.5(iii), if $i \neq j$, $L_i \otimes_k X \not\cong L_j \otimes_k X$, then $(\bigoplus_{i=1}^r L_i \otimes_k X) \mid kQ\#k\langle\sigma^m\rangle \otimes_{kQ} X$ by Krull-Schmidt Theorem. And $kQ\#k\langle\sigma^m\rangle \otimes_{kQ} X \cong \bigoplus_{i=1}^r L_i \otimes_k X$ since $kQ\#k\langle\sigma^m\rangle \otimes_{kQ} X$ has exactly r indecomposable summands by Lemma 4.3(v).

(ii) For a $kQ\#k\langle\sigma^m\rangle$ -module Y , $Y \cong X$ as kQ -modules, then Y is an indecomposable $kQ\#k\langle\sigma^m\rangle$ -module and by Lemma 4.4, $Y \mid kQ\#k\langle\sigma^m\rangle \otimes_{kQ} Y \cong kQ\#k\langle\sigma^m\rangle \otimes_{kQ} X$. Then by (i), Lemma 4.5(iii) and the Krull-Schmidt Theorem, there exists a unique $i \in \{1, 2, \dots, r\}$, such that $Y \cong L_i \otimes_k X$. ■

4.2 Induction of indecomposable $kQ\#k\langle\sigma\rangle$ -modules from an indecomposable $\langle\sigma\rangle$ -equivalent kQ -module

In this section, we are ready to answer Question 2.

Theorem 4.8 *Let X be an indecomposable kQ -module with m minimal such that $\sigma^m X \cong X$, L_i as defined in the isomorphism relation (1), for any $i \in \{1, 2, \dots, r\}$. Then the following statements hold:*

- (i) $(kQ\#k\langle\sigma\rangle) \otimes_{kQ\#k\langle\sigma^m\rangle} (L_i \otimes_k X) \cong X \oplus \sigma X \oplus \dots \oplus \sigma^{m-1} X$ as kQ -modules;
- (ii) $(kQ\#k\langle\sigma\rangle) \otimes_{kQ\#k\langle\sigma^m\rangle} (L_i \otimes_k X)$ is an indecomposable $kQ\#k\langle\sigma\rangle$ -module;
- (iii) $(kQ\#k\langle\sigma\rangle) \otimes_{kQ\#k\langle\sigma^m\rangle} (L_i \otimes_k X) \not\cong (kQ\#k\langle\sigma\rangle) \otimes_{kQ\#k\langle\sigma^m\rangle} (L_j \otimes_k X)$ as $kQ\#k\langle\sigma\rangle$ -module, if $i \neq j$;
- (iv) $(kQ\#k\langle\sigma\rangle) \otimes_{kQ} X$ is isomorphic to the direct sum of r non-isomorphic $kQ\#k\langle\sigma\rangle$ -modules, that is, $(kQ\#k\langle\sigma\rangle) \otimes_{kQ} X \cong \bigoplus_{i=1}^r (kQ\#k\langle\sigma\rangle) \otimes_{kQ\#k\langle\sigma^m\rangle} (L_i \otimes_k X)$ as $kQ\#k\langle\sigma\rangle$ -modules;
- (v) For any $kQ\#k\langle\sigma\rangle$ -module Y , if $Y \cong X \oplus \sigma X \oplus \dots \oplus \sigma^{m-1} X$ as kQ -modules, then there exists a unique $i \in \{1, 2, \dots, r\}$, such that $Y \cong (kQ\#k\langle\sigma\rangle) \otimes_{kQ\#k\langle\sigma^m\rangle} (L_i \otimes_k X)$. That is, there are r non-isomorphic $kQ\#k\langle\sigma\rangle$ -modules induced from the indecomposable $\langle\sigma\rangle$ -equivalent kQ -modules $X \oplus \sigma X \oplus \dots \oplus \sigma^{m-1} X$.

Proof. (i) $(kQ\#k\langle\sigma\rangle) \otimes_{kQ\#k\langle\sigma^m\rangle} (L_i \otimes_k X) = 1 \otimes X \oplus \sigma \otimes X \dots \oplus \sigma^{m-1} \otimes X \cong X \oplus \sigma X \oplus \dots \oplus \sigma^{m-1} X$ as kQ -modules since $\sigma^j X \cong \sigma^j \otimes X$ as kQ -modules. In fact, define $f : \sigma^j X \rightarrow \sigma^j \otimes X$ by $f(x) = \sigma^j \otimes x$, for any $x \in X$. Then f is bijection, and f is a kQ -module homomorphism since $f(p \cdot x) = \sigma^j \otimes \sigma^{-j}(p)(x) = \sigma^j(\sigma^{-j}(p)) \otimes x = p(\sigma^j \otimes x) = pf(x)$, $\forall p \in kQ, x \in X$.

(ii) $(kQ\#k\langle\sigma\rangle) \otimes_{kQ\#k\langle\sigma^m\rangle} (L_i \otimes_k X)$ is an indecomposable $kQ\#k\langle\sigma\rangle$ -module, since $kQ\#k\langle\sigma\rangle \otimes_{kQ\#k\langle\sigma^m\rangle} (L_i \otimes_k X)$ is an indecomposable $\langle\sigma\rangle$ -equivalent kQ -module by (i) and Lemma 3.1.

(iii) For any $i \in \{1, 2, \dots, r\}$, $(kQ \# k\langle\sigma\rangle) \otimes_{kQ \# k\langle\sigma^m\rangle} (L_i \otimes_k X) = 1 \otimes L_i \otimes_k X \oplus \sigma \otimes L_i \otimes_k X \cdots \oplus \sigma^{m-1} \otimes L_i \otimes_k X$ as $kQ \# k\langle\sigma^m\rangle$ -modules. Otherwise, if $i \neq j$, $kQ \# k\langle\sigma\rangle \otimes_{kQ \# k\langle\sigma^m\rangle} (L_i \otimes_k X) \cong kQ \# k\langle\sigma\rangle \otimes_{kQ \# k\langle\sigma^m\rangle} (L_j \otimes_k X)$ as $kQ \# k\langle\sigma\rangle$ -modules. Then $1 \otimes L_i \otimes_k X \oplus \sigma \otimes L_i \otimes_k X \cdots \oplus \sigma^{m-1} \otimes L_i \otimes_k X \cong 1 \otimes L_j \otimes_k X \oplus \sigma \otimes L_j \otimes_k X \cdots \oplus \sigma^{m-1} \otimes L_j \otimes_k X$ as $kQ \# k\langle\sigma^m\rangle$ -modules, which is a contradiction since for $1 \leq s \leq m-1$, $1 \otimes L_i \otimes_k X \cong X \not\cong \sigma^s X \cong \sigma^s \otimes L_j \otimes_k X$ as kQ -modules and $1 \otimes L_i \otimes_k X \cong L_i \otimes_k X \not\cong L_j \otimes_k X \cong 1 \otimes L_j \otimes_k X$ as $kQ \# k\langle\sigma^m\rangle$ -modules.

(iv) For any $i \in \{1, 2, \dots, r\}$, we have $kQ \# k\langle\sigma\rangle \otimes_{kQ \# k\langle\sigma^m\rangle} (L_i \otimes_k X) \mid kQ \# k\langle\sigma\rangle \otimes_{kQ \# k\langle\sigma\rangle} (L_i \otimes_k X)$ by Lemma 4.4, $kQ \# k\langle\sigma\rangle \otimes_{kQ \# k\langle\sigma^m\rangle} (L_i \otimes_k X) \mid kQ \# k\langle\sigma\rangle \otimes_{kQ} (X \oplus \sigma X \oplus \cdots \oplus \sigma^{m-1} X)$ by (i), and $kQ \# k\langle\sigma\rangle \otimes_{kQ \# k\langle\sigma^m\rangle} (L_i \otimes_k X) \mid kQ \# k\langle\sigma\rangle \otimes_{kQ} X$ by (ii) and Lemma 4.3(ii). By (iii) and the Krull-Schmidt Theorem, we have $(\oplus_{i=1}^r kQ \# k\langle\sigma\rangle \otimes_{kQ \# k\langle\sigma^m\rangle} (L_i \otimes_k X)) \mid kQ \# k\langle\sigma\rangle \otimes_{kQ} X$. Additionally $kQ \# k\langle\sigma\rangle \otimes_k X \cong \oplus_{i=1}^r kQ \# k\langle\sigma\rangle \otimes_{kQ \# k\langle\sigma^m\rangle} (L_i \otimes_k X)$ since $kQ \# k\langle\sigma\rangle \otimes_{kQ} X$ has exactly r indecomposable summands by Lemma 4.3(v).

(v) For a $kQ \# k\langle\sigma\rangle$ -module Y , $Y \cong X \oplus \sigma X \oplus \cdots \oplus \sigma^{m-1} X$ as kQ -modules, then Y is an indecomposable $kQ \# k\langle\sigma\rangle$ -module and by Lemma 4.4, $Y \mid kQ \# k\langle\sigma\rangle \otimes_{kQ} Y \cong kQ \# k\langle\sigma\rangle \otimes_{kQ} (X \oplus \sigma X \oplus \cdots \oplus \sigma^{m-1} X)$, then by (iv), Lemma 4.3(ii) and the Krull-Schmidt Theorem, there exists a unique $i \in \{1, 2, \dots, r\}$, such that $Y \cong kQ \# k\langle\sigma\rangle \otimes_{kQ \# k\langle\sigma^m\rangle} L_i \otimes_k X$. ■

Theorem 4.9 *Any indecomposable $kQ \# k\langle\sigma\rangle$ -module is an indecomposable $\langle\sigma\rangle$ -equivalent kQ -module. Conversely, for any indecomposable $\langle\sigma\rangle$ -equivalent kQ -module, the corresponding canonical induced $kQ \# k\langle\sigma\rangle$ -module is indecomposable.*

Proof. Given any indecomposable $kQ \# k\langle\sigma\rangle$ -module X , by Lemma 2.3, $X \cong \oplus_{j=1}^s X_j$, $X_j \cong Y_j \oplus \sigma Y_j \oplus \cdots \oplus \sigma^{m_j-1} Y_j$ with Y_j an indecomposable kQ -module and m_j minimal such that $\sigma^{m_j} Y_j \cong Y_j$. By Lemma 4.4, we have $X \mid kQ \# k\langle\sigma\rangle \otimes_{kQ} X \cong \oplus_{j=1}^s \oplus_{k=0}^{m_j-1} kQ \# k\langle\sigma\rangle \otimes_{kQ} \sigma^k Y_j$, then by Lemma 4.3(ii) and the Krull-Schmidt Theorem, there exists j , such that $X \mid kQ \# k\langle\sigma\rangle \otimes_{kQ} Y_j$. Thus by Theorem 4.8, we have $X \cong Y_j \oplus \sigma Y_j \oplus \cdots \oplus \sigma^{m_j-1} Y_j$ as kQ -modules, that is, X is an indecomposable $\langle\sigma\rangle$ -equivalent kQ -modules.

Conversely, since any $kQ \# k\langle\sigma\rangle$ -module is a $\langle\sigma\rangle$ -equivalent kQ -module. ■

According to Theorem 4.8 and Theorem 4.9, our main purpose has been carried out, that is, all indecomposable $kQ \# k\langle\sigma\rangle$ -modules can be constructed from indecomposable kQ -modules as follows:

(I) For a fixed indecomposable kQ -module X , write m to be the minimal positive integer satisfying $\sigma^m X \cong X$. On the indecomposable $\langle\sigma\rangle$ -equivalent kQ -module $Y = X \oplus \sigma X \oplus \cdots \oplus \sigma^{m-1} X$, there are induced $r = n/m$ indecomposable $kQ \# k\langle\sigma\rangle$ -modules,

which are $(kQ \# k\langle\sigma\rangle) \otimes_{kQ \# k\langle\sigma^m\rangle} (L_i \otimes_k X)$, $i = 1, \dots, r$;

(II) For any indecomposable $kQ \# k\langle\sigma\rangle$ -module Y , there exists an indecomposable kQ -module X , so then apply (I), and there exists a unique $j \in \{1, 2, \dots, r\}$, such that $Y \cong (kQ \# k\langle\sigma\rangle) \otimes_{kQ \# k\langle\sigma^m\rangle} (L_j \otimes_k X)$.

We end the section by giving the relation between simple, projective and injective modules between $\text{mod } kQ$ and $\text{mod } kQ \# k\langle\sigma\rangle$.

Lemma 4.10 *Let H be a finite dimensional semisimple Hopf algebra and A a finite dimensional H -module algebra. For a left $A \# H$ -module I , if I is an injective A -module, then I is an injective $A \# H$ -module.*

Proof. For an $A \# H$ -module M, N , let $g : M \rightarrow N$ and $h : M \rightarrow I$ be two $A \# H$ -module homomorphisms such that g is injective. In order to prove that I is injective as an $A \# H$ -module, it is enough to find an $\tilde{f} \in \text{Hom}_{A \# H}(N, I)$ satisfying $h = \tilde{f}g$. Since I is injective as an A -module, there is an $f \in \text{Hom}_A(N, I)$ such that $h = fg$, where we consider $A \# H$ -modules as A -modules in the natural way. Define $\tilde{f}(n) = \sum_{(t)} S(t_1) \cdot f(t_2 \cdot n)$ for $n \in N$, where t is a non-zero right integral with $\varepsilon(t) = 1$. Then \tilde{f} is $A \# H$ -linear by Proposition 2 in [3], and $h = \tilde{f}g$ since $\tilde{f}g(m) = \sum_{(t)} S(t_1) \cdot f(t_2 \cdot g(m)) = \sum_{(t)} S(t_1) \cdot f(g(t_2 \cdot m)) = \sum_{(t)} S(t_1) \cdot fg(t_2 \cdot m) = \sum_{(t)} S(t_1) \cdot h(t_2 \cdot m) = \sum_{(t)} S(t_1) \cdot (t_2 \cdot h(m)) = (\sum_{(t)} S(t_1)t_2) \cdot h(m) = \varepsilon(t)h(m) = h(m)$. ■

Recall that $t \in H$ is a non-zero right integral, if $th = \varepsilon(h)t$, for any $h \in H$. Since H is semisimple Hopf algebra, there must exist a non-zero right integral such that $\varepsilon(t) = 1$. For details, see [13].

Theorem 4.11 *Let X be a $kQ \# k\langle\sigma\rangle$ -module, then:*

- (i) *X is simple if and only if there exists a simple kQ -module S , such that X is isomorphic to one of the $kQ \# k\langle\sigma\rangle$ -modules induced from the indecomposable $\langle\sigma\rangle$ -equivalent kQ -module $\oplus_{i=1}^{m-1} \sigma^i S$.*
- (ii) *X is projective if and only if there exists an indecomposable projective kQ -module P , such that X is isomorphic to one of the $kQ \# k\langle\sigma\rangle$ -modules induced from the indecomposable $\langle\sigma\rangle$ -equivalent kQ -module $\oplus_{i=1}^{m-1} \sigma^i P$.*
- (iii) *X is injective if and only if there exists an indecomposable injective kQ -module I , such that X is isomorphic to one of the $kQ \# k\langle\sigma\rangle$ -modules induced from the indecomposable $\langle\sigma\rangle$ -equivalent kQ -module $\oplus_{i=1}^{m-1} \sigma^i I$.*

Proof. According to Theorem 4.8 and Theorem 4.9, we need only to prove that for a $kQ\#k\langle\sigma\rangle$ -module, X is a semisimple (projective, injective) $kQ\#k\langle\sigma\rangle$ -module if and only if X is a semisimple (projective, injective) kQ -module.

(i) Any $kQ\#k\langle\sigma\rangle$ -module X induced from an indecomposable $\langle\sigma\rangle$ -equivalent kQ -module $\bigoplus_{i=1}^{m-1} \sigma^i S$ is a simple $kQ\#k\langle\sigma\rangle$ -module since the dimension of vector space X_i is 0 or 1, for any $i \in Q_0$. Additionally, any simple $kQ\#k\langle\sigma\rangle$ -module X is a semisimple kQ -module as in [14].

(ii) By Lemma 3.1.7 in [8], for a $kQ\#k\langle\sigma\rangle$ -module, X is a projective $kQ\#k\langle\sigma\rangle$ -module if and only if X is a projective kQ -module.

(iii) By Lemma 4.10, for a $kQ\#k\langle\sigma\rangle$ -module, X is an injective $kQ\#k\langle\sigma\rangle$ -module if and only if X is an injective kQ -module.

5 Applications

In this section, we apply the results we have gotten by giving some examples. Let ξ be the n -th primitive root of 1.

Example 5.1 Given a quiver Q , $\sigma \in \text{Aut}(Q)$ of order n . In this example, we are going to construct all simple $kQ\#k\langle\sigma\rangle$ -modules in a concrete way. Let S be a simple kQ -module with m minimal such that $\sigma^m S \cong S$ (in fact $\sigma^m S = S$ since S is simple), and let $r = n/m$.

Let $\mathcal{S}^{(l)}, l \in \{0, 1, \dots, r-1\}$, as kQ -module, is an indecomposable $\langle\sigma\rangle$ -equivalent kQ -module $S \oplus \sigma S \oplus \dots \oplus \sigma^{m-1} S$, and σ 's action on $\mathcal{S}^{(l)} = S \oplus \sigma S \oplus \dots \oplus \sigma^{m-1} S$ is defined by

$$\sigma(x_0, x_1, \dots, x_{m-1}) = (\xi^{ml} x_{m-1}, x_0, \dots, x_{m-2}), x_i \in S.$$

Claim 1: $p(x) = \sigma^{-m}(p)(x)$, for any $x \in S$.

Proof. For a simple kQ -module S , there exists a unique $i \in Q_0$, such that for any $x \in S, q \in Q_0 \cup Q_1 \setminus \{e_i\}$, $e_i(x) = x, q(x) = 0$. Since $\sigma^m S = S$, then $\sigma^m(e_i) = e_i, \sigma^q \in Q_0 \cup Q_1 \setminus \{e_i\}$, for any $q \in Q_0 \cup Q_1 \setminus \{e_i\}$. ■

Claim 2: For any $l \in \{0, 1, \dots, r-1\}$, $\mathcal{S}^{(l)}$ is a $kQ\#k\langle\sigma\rangle$ -module.

Proof. We need only to prove that two equations $\sigma^n = 1$ and $p\#\sigma = \sigma\sigma^{-1}(p)$ are satisfied as actions on $\mathcal{S}^{(l)}$:

$$\begin{aligned} \sigma^n(x_0, x_1, \dots, x_{m-1}) &= \xi^{mr}(x_0, x_1, \dots, x_{m-1}) \\ &= (x_0, x_1, \dots, x_{m-1}), \\ (p\#\sigma)(x_0, x_1, \dots, x_{m-1}) &= p(\xi^{ml} x_{m-1}, x_0, \dots, x_{m-2}) \\ &= (\xi^{ml} p(x_{m-1}), \sigma^{-1}(p)(x_0), \dots, \sigma^{-(m-1)}(p)(x_{m-2})) \end{aligned}$$

$$\begin{aligned}
&= \sigma(\sigma^{-1}(p)(x_0), \sigma^{-2}(p)(x_1), \dots, p(x_{m-1})) \\
&\stackrel{\text{Claim 1}}{=} \sigma(\sigma^{-1}(p)(x_0), \sigma^{-2}(p)(x_1), \dots, \sigma^{-m}(p)(x_{m-1})) \\
&= (\sigma\sigma^{-1}(p))(x_0, x_1, \dots, x_{m-1}). \quad \blacksquare
\end{aligned}$$

Claim 3 : If $l_1 \neq l_2 \in \{0, 1, \dots, r-1\}$, then $\mathcal{S}^{(l_1)} \not\cong \mathcal{S}^{(l_2)}$ as $kQ \# k\langle \sigma \rangle$ -modules.

Proof. Simply let $l_1 = 0, l_2 = l$, for some $l \in \{1, 2, \dots, r-1\}$. Otherwise, there exists a $kQ \# k\langle \sigma \rangle$ -isomorphism $F : \mathcal{S}^{(0)} \rightarrow \mathcal{S}^{(l)}$, denoted $F(x, 0, \dots, 0) = (F(x)_0, F(x)_1, \dots, F(x)_{m-1})$.

$$\begin{aligned}
F(\sigma(0, \dots, 0, x)) &= F(x, 0, \dots, 0) \\
&= (F(x)_0, F(x)_1, \dots, F(x)_{m-1}), \\
\sigma(F(0, \dots, 0, x)) &= \sigma(F(\sigma^{m-1}(x, 0, \dots, 0))) \\
&= \sigma^m F(x, 0, \dots, 0) \\
&= \sigma^m (F(x)_0, F(x)_1, \dots, F(x)_{m-1}) \\
&= \xi^{ml} (F(x)_0, F(x)_1, \dots, F(x)_{m-1}).
\end{aligned}$$

From $F\sigma = \sigma F$, we get $\xi^{ml} = 1$, which is contradicted since $l \in \{1, 2, \dots, r-1\}$ and ξ is a primitive root of 1. \blacksquare

So $\{\mathcal{S}^{(0)}, \mathcal{S}^{(1)}, \dots, \mathcal{S}^{(r-1)}\}$ are exactly r non-isomorphic $kQ \# k\langle \sigma \rangle$ -modules induced from an indecomposable $\langle \sigma \rangle$ -equivalent kQ -module $S \oplus {}^\sigma S \oplus \dots \oplus {}^{\sigma^{m-1}} S$.

Example 5.2 *Given a quiver Q , $\sigma \in \text{Aut}(Q)$ of order n . In this example, we are going to construct all indecomposable projective $kQ \# k\langle \sigma \rangle$ -modules in a concrete way. Let P be an indecomposable kQ -module with m minimal such that $\sigma^m P \cong P$, and let $r = n/m$.*

Let $\mathcal{P}^{(l)}, l \in \{0, 1, \dots, r-1\}$, as kQ -module, is an indecomposable $\langle \sigma \rangle$ -equivalent kQ -module $P \oplus {}^\sigma P \oplus \dots \oplus {}^{\sigma^{m-1}} P$, and σ 's action on $\mathcal{P}^{(l)} = P \oplus {}^\sigma P \oplus \dots \oplus {}^{\sigma^{m-1}} P$ is defined by

$$\sigma(x_0, x_1, \dots, x_{m-1}) = (\xi^{ml} \sigma^m(x_{m-1}), x_0, \dots, x_{m-2}), x_i \in P.$$

Claim 1 : The action of σ is well-defined due to $\sigma^m(x) \in P$ for any $x \in P$.

Proof. For an indecomposable projective kQ -module P , there exists a unique $i \in Q_0$, such that $P = kQe_i$. Since kQ is a $k\langle \sigma \rangle$ -module algebra, we have $\sigma^m(P) = \sigma^m(kQ)\sigma^m(e_i) = kQ\sigma^m(e_i)$, then we need only to prove $\sigma^m(e_i) = e_i$. It is clearly true by considering the simple module $S = P/rP$. \blacksquare

Claim 2 : For any $l \in \{0, 1, \dots, r-1\}$, $\mathcal{P}^{(l)}$ is a $kQ \# k\langle \sigma \rangle$ -module.

Proof. We need only to prove that two equation $\sigma^n = 1$ and $p\#\sigma = \sigma\sigma^{-1}(p)$, are satisfied as actions on $\mathcal{S}^{(l)}$:

$$\sigma^n(x_0, x_1, \dots, x_{m-1}) = \xi^{mr}(\sigma^{mr}(x_0), \sigma^{mr}(x_1), \dots, \sigma^{mr}(x_{m-1}))$$

$$\begin{aligned}
&= (x_0, x_1, \dots, x_{m-1}), \\
(p\#\sigma)(x_0, x_1, \dots, x_{m-1}) &= p(\xi^{ml}\sigma^m(x_{m-1}), x_0, \dots, x_{m-2}) \\
&= (\xi^{ml}p\sigma^m(x_{m-1}), \sigma^{-1}(p)(x_0), \dots, \sigma^{-(m-1)}(p)(x_{m-2})) \\
&= (\xi^{ml}\sigma^m\sigma^{-m}(p)(x_{m-1}), \sigma^{-1}(p)(x_1), \dots, \sigma^{-(m-1)}(p)(x_{m-2})) \\
&= \sigma(\sigma^{-1}(p)(x_0), \sigma^{-2}(p)(x_1), \dots, \sigma^{-m}(p)(x_{m-1})) \\
&= (\sigma\sigma^{-1}(p))(x_0, x_1, \dots, x_{m-1}). \quad \blacksquare
\end{aligned}$$

Claim 3 : If $l_1 \neq l_2 \in \{0, 1, \dots, r-1\}$, then $\mathcal{P}^{(l_1)} \not\cong \mathcal{P}^{(l_2)}$ as $kQ\#k\langle\sigma\rangle$ -modules.

Proof. Simply we let $l_1 = 0, l_2 = l$, for some $l \in \{1, 2, \dots, r-1\}$. Otherwise, there exists a $kQ\#k\langle\sigma\rangle$ -isomorphism $F : \mathcal{P}^{(0)} \rightarrow \mathcal{P}^{(l)}$, denoted $F(x, 0, \dots, 0) = (F(x)_0, F(x)_1, \dots, F(x)_{m-1})$.

$$\begin{aligned}
F(\sigma(0, \dots, 0, x)) &= F(\sigma^m(x), 0, \dots, 0) \\
&= (F(\sigma^m(x))_0, F(\sigma^m(x))_1, \dots, F(\sigma^m(x))_{m-1}), \\
\sigma(F(0, \dots, 0, x)) &= \sigma(F(\sigma^{m-1}(x, 0, \dots, 0))) \\
&= \sigma^m F(x, 0, \dots, 0) \\
&= \sigma^m(F(x)_0, F(x)_1, \dots, F(x)_{m-1}) \\
&= \xi^{ml}(\sigma^m(F(x)_0), \sigma^m(F(x)_1), \dots, \sigma^m(F(x)_{m-1})).
\end{aligned}$$

From $F\sigma = \sigma F$, particularly, $F\sigma(e_i) = \sigma F(e_i)$, we get $\xi^{ml} = 1$, which is contradicted since $l \in \{1, 2, \dots, r-1\}$ and ξ is a primitive root of 1. \blacksquare

So $\{\mathcal{P}^{(0)}, \mathcal{P}^{(1)}, \dots, \mathcal{P}^{(r-1)}\}$ are the r non-isomorphic $kQ\#k\langle\sigma\rangle$ -modules induced from an indecomposable $\langle\sigma\rangle$ -equivalent kQ -module $P \oplus {}^\sigma P \oplus \dots \oplus {}^{\sigma^{m-1}} P$.

Example 5.3 *Let Q be the Kronecker quiver*

$$\begin{array}{ccc}
& \xleftarrow{\alpha_1} & \\
a_0 & & a_1 \\
& \xleftarrow{\alpha_0} &
\end{array}$$

Let $p(l), i(l), l \in \mathbf{N}, r_\lambda(l), r_\infty(l), l \in \mathbf{N} \setminus \{0\}, \lambda \in k$ be the kQ -modules defined by

$$\begin{array}{ccc}
p(l) : & k^{l+1} \xleftarrow{\begin{bmatrix} \mathbf{I}_l \\ 0 \end{bmatrix}} k^l & i(l) : & k^l \xleftarrow{\begin{bmatrix} \mathbf{I}_l, 0 \\ 0, \mathbf{I}_l \end{bmatrix}} k^{l+1} \\
r_\lambda(l) : & k^l \xleftarrow{\begin{matrix} J_\lambda(l) \\ \mathbf{I}_l \end{matrix}} k^l & r_\infty(l) : & k^l \xleftarrow{\begin{matrix} \mathbf{I}_l \\ J_0(l) \end{matrix}} k^l
\end{array}$$

where $J_\lambda(l), \lambda \in k$ is the $l \times l$ Jordan block

$$\begin{pmatrix} \lambda & 1 & & \\ & \ddots & \ddots & \\ & & \lambda & 1 \\ & & & \lambda \end{pmatrix}$$

It is known in [4, 7] that $\{p(l), i(l), l \in \mathbf{N}, r_\lambda(l), r_\infty(l), l \in \mathbf{N} \setminus \{0\}, \lambda \in k\}$ classify all indecomposable kQ -modules up to isomorphism.

Let $P(l)^{(0)}, l \in \mathbf{N}$, be the $kQ \# k\langle \sigma \rangle$ -module that is $p(l)$ as kQ -module and σ' 's action is defined by

$$\left(\begin{pmatrix} & & 1 \\ & \cdots & \\ & 1 & \\ 1 & & \end{pmatrix}_{l+1, l+1}, \begin{pmatrix} & & 1 \\ & \cdots & \\ & 1 & \\ 1 & & \end{pmatrix}_{l, l} \right).$$

Let $P(l)^{(1)}, l \in \mathbf{N}$, be the $kQ \# k\langle \sigma \rangle$ -module that is $p(l)$ as kQ -module and σ' 's action is defined by

$$\left(- \begin{pmatrix} & & 1 \\ & \cdots & \\ & 1 & \\ 1 & & \end{pmatrix}_{l+1, l+1}, - \begin{pmatrix} & & 1 \\ & \cdots & \\ & 1 & \\ 1 & & \end{pmatrix}_{l, l} \right).$$

Let $I(l)^{(0)}, l \in \mathbf{N}$, be the $kQ \# k\langle \sigma \rangle$ -module that is $i(l)$ as kQ -module and σ' 's action is defined by

$$\left(\begin{pmatrix} & & 1 \\ & \cdots & \\ & 1 & \\ 1 & & \end{pmatrix}_{l, l}, \begin{pmatrix} & & 1 \\ & \cdots & \\ & 1 & \\ 1 & & \end{pmatrix}_{l+1, l+1} \right).$$

Let $I(l)^{(1)}, l \in \mathbf{N}$, be the $kQ \# k\langle \sigma \rangle$ -module that is $i(l)$ as kQ -module and σ' 's action is defined by

$$\left(- \begin{pmatrix} & & 1 \\ & \cdots & \\ & 1 & \\ 1 & & \end{pmatrix}_{l, l}, - \begin{pmatrix} & & 1 \\ & \cdots & \\ & 1 & \\ 1 & & \end{pmatrix}_{l+1, l+1} \right).$$

Let $R_{(0,\infty)}(l), l \in \mathbf{N} \setminus \{0\}$, be the $kQ \# k\langle\sigma\rangle$ -module that is $r_0(l) \oplus r_\infty(l)$ as kQ -modules and σ' 's action is defined by

$$\sigma((x_0, x_1), (y_0, y_1)) = ((y_0, y_1), (x_0, x_1)), \forall x_0, x_1, y_0, y_1 \in k^l.$$

Let $R_{(\lambda, \lambda^{-1})}(l), \lambda \in k \setminus \{0\}, l \in \mathbf{N} \setminus \{0\}$, be the $kQ \# k\langle\sigma\rangle$ -module that is $r_\lambda(l) \oplus r_{\lambda^{-1}}(l)$ as kQ -modules and σ' 's action is defined by

$$\sigma((x_0, x_1), (y_0, y_1)) = ((B_l^{-1}(y_0), A_l^{-1}(y_1)), (B_l(x_0), A_l(x_1))), \forall x_0, x_1, y_0, y_1 \in k^l,$$

where $B_l = (b_{ij})_{l \times l}, A_l = (a_{ij})_{l \times l} \in M_{l \times l}(k)$ satisfy

$$\begin{aligned} b_{ij} &= 0, \quad i > j, \\ b_{il} &= 0, \quad 1 \leq i < l, \\ b_{ll} &= 1, \\ b_{ij} &= -(b_{i-1,j}\lambda + b_{i-1,j+1}\lambda^2), \quad 1 \leq i < l, i \leq j < l. \\ a_{ij} &= 0, \quad i > j, \\ a_{ll} &= \lambda, \\ a_{ij} &= -(a_{i-1,j}\lambda + a_{i-1,j+1}\lambda^2), \quad 1 \leq i < l, i \leq j < l, \\ a_{il} &= -a_{i-1,l}\lambda, \quad 1 \leq i < l. \end{aligned}$$

It is easy to see that

$$\{P(l)^{(0)}, P(l)^{(1)}, I(l)^{(0)}, I(l)^{(1)}, l \in \mathbf{N}, R_{(0,\infty)}(l), R_{(\lambda, \lambda^{-1})}(l), l \in \mathbf{N} \setminus \{0\}, \lambda \in k \setminus \{0\}\}$$

classify all indecomposable $kQ \# k\langle\sigma\rangle$ -modules up to isomorphism.

Acknowledgement. The paper was ultimately finished during the first author's visit at Kansas State University. The first author thanks The China Council Scholarship for the financial support of her visit to Kansas State University. The authors thank Professor Zongzhu Lin of Kansas State University for his discussions and suggestions.

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